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BUCKLING OF AN AXIALLY COMPRESSED CYLINDRICAL SHELL OF VARIABLE THICKNESS

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Abstract—This paper focuses on the buckling of cylindrical shells with small thickness variations. Two important cases of thickness variation pattern are considered. Asymptotic formulas up to the second order of the thickness variation parameter ε are derived by the combination of the perturbation and weighted residual methods. The expressions obtained in this study reduce to Koiter's formulas, when only the first-order term of the thickness variation parameter is retained in the analysis. Results from the asymptotic formulas are compared with those obtained through the purely numerical techniques of the finite difference method and the shooting method.

1. INTRODUCTION

There is a vast literature devoted to buckling of cylindrical shells of constant thickness. The problem of the influence of thickness variation on the buckling load has not gained attention, and remains open even today. To the best of our knowledge, the first work on the effect of thickness variation on the buckling of shells was undertaken by the present authors (Elishakoff *et al.*, 1992). Both the thickness variation and the initial geometric imperfections were considered axisymmetrical. The solution was composed of two terms; the first being associated with the shell of constant thickness, whereas the second incorporated the effects of the thickness variation. The former coincided with Koiter's analytical investigation (1992) for constant thickness shells with axisymmetric imperfection, whereas the latter term was derived numerically using the shooting method. In Koiter (1963), an analytical formula has been derived for the buckling load of a perfect, non-uniform cylindrical shell. The attendant derivation through utilizing the energy method was included in Elishakoff and Charmats (1977). Elishakoff and Charmats supported the central result of the combined theoretical–numerical investigation that the effect of thickness variation becomes remarkable when the thickness pattern is co-configurational with the initial imperfection. However, further investigation shows the effect of the axisymmetric thickness variation occurs at twice the wave number of the classical buckling mode.

The present study examines in detail the buckling of the cylindrical shell with small thickness variations. Our analysis is based on a system of linearized governing differential equations of perfect shells with variable thickness. Asymptotic formulas in terms of ε (ε is the thickness non-uniformity parameter), are derived by a hybrid perturbation-weighted residuals method. In comparison with formulas (Koiter, 1992, 1993), which are linear in ε, these asymptotic formulas also contain the quadratic term, which results in a higher accuracy. In addition to the analytic investigation, numerical study is also performed, and results stemming from different methods were compared and discussed.

2. BASIC EQUATIONS

The linear equations governing the axially compressed, non-uniform cylindrical shell are as follows:

$$h^2\nabla^2\nabla^2 F+2\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2\left(\frac{\partial^2 F}{\partial x^2}-\nu\frac{\partial^2 F}{\partial y^2}\right)-h\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}\left(\frac{\partial^2 F}{\partial x^2}-\nu\frac{\partial^2 F}{\partial y^2}\right)$$

$$-2h\frac{\mathrm{d}h}{\mathrm{d}x}\left(\frac{\partial^3 F}{\partial x^3}-\nu\frac{\partial^3 F}{\partial x\partial y^2}\right)-2(1+\nu)h\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\partial^3 F}{\partial x\partial y^2}=\frac{Eh^3}{R}\frac{\partial^2 W}{\partial x^2} \quad (1)$$

$$\frac{Eh^3}{12(1-\nu^2)}\nabla^2\nabla^2 W+\frac{1}{R}\frac{\partial^2 F}{\partial x^2}+\frac{3Eh^2}{12(1-\nu^2)}\frac{\mathrm{d}h}{\mathrm{d}x}\nabla^2 W+\frac{3Eh^2}{12(1-\nu^2)}\left(\frac{\partial^3 W}{\partial x^3}+\nu\frac{\partial^3 W}{\partial x\partial y^2}\right)\frac{\mathrm{d}h}{\mathrm{d}x}$$

$$+\frac{6Eh}{12(1-\nu^2)}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial^2 y}\right)+\frac{3Eh^2}{12(1-\nu^2)}\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}$$

$$+\frac{3Eh^2}{12(1+\nu)}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\partial^3 W}{\partial x\partial y^2}+P_0\frac{\partial^2 W}{\partial x^2}=0, \quad (2)$$

where W and F represent the radial displacement (positive outward) and the Airy stress function, respectively; ν is Poisson's ratio, E the modulus of elasticity; P_0 denotes the uniform axial load at the ends of the shell; $h(x)$ is the shell thickness, assumed here varying only axisymmetrically

$$h(x)=h_0\left(1-\varepsilon\cos\frac{2px}{R}\right), \quad (3)$$

where h_0 is the nominal thickness of the shell; ε and p are the non-dimensional parameters indicating the magnitude and wave of the thickness variation.

By introducing the following non-dimensional parameters

$$\xi=\frac{x}{L},\quad \eta=\frac{y}{L},\quad w=\frac{W}{L},\quad f=\frac{F}{D},\quad H=\frac{h}{h_0} \quad (4)$$

the governing equations (1) and (2) can be rewritten into their non-dimensional form

$$H^2\bar{\nabla}^2\bar{\nabla}^2 f+2\left(\frac{\mathrm{d}H}{\mathrm{d}\xi}\right)^2\left(\frac{\partial^2 f}{\partial\xi^2}-\nu\frac{\partial^2 f}{\partial\eta^2}\right)-H\frac{\partial^2 H}{\partial\xi^2}\left(\frac{\partial^2 f}{\partial\xi^2}-\nu\frac{\partial^2 f}{\partial\eta^2}\right)$$

$$-2H\frac{\mathrm{d}H}{\mathrm{d}\xi}\left(\frac{\partial^3 f}{\partial\xi^3}-\nu\frac{\partial^3 f}{\partial\xi\partial\eta^2}\right)-2(1+\nu)H\frac{\mathrm{d}H}{\mathrm{d}\xi}\frac{\partial^3 f}{\partial\xi\partial\eta^2}=\frac{12(1-\nu^2)L^2}{Rh_0}H^3\frac{\partial^2 w}{\partial\xi^2}, \quad (5)$$

$$H^3\bar{\nabla}^2\bar{\nabla}^2 w+\frac{P_0L^2}{D_0}\frac{\partial^2 w}{\partial\xi^2}+\frac{L^2}{Rh_0}\frac{\partial^2 f}{\partial\xi^2}+3H^2\frac{\mathrm{d}H}{\mathrm{d}\xi}\frac{\partial}{\partial\xi}\bar{\nabla}^2 w+3H^2\frac{\mathrm{d}H}{\mathrm{d}\xi}\left(\frac{\partial^3 w}{\partial\xi^3}+\nu\frac{\partial^3 w}{\partial\xi\partial\eta^2}\right)$$

$$+6H\left(\frac{\mathrm{d}H}{\mathrm{d}\xi}\right)^2\left(\frac{\partial^2 w}{\partial\xi^2}+\nu\frac{\partial^2 w}{\partial\eta^2}\right)+3H^2\left(\frac{\partial^2 w}{\partial\xi^2}+\nu\frac{\partial^2 w}{\partial\eta^2}\right)\frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2}$$

$$+3(1-\nu^2)H^2\frac{\mathrm{d}H}{\mathrm{d}\xi}\frac{\partial^3 w}{\partial\xi\partial\eta^2}=0. \quad (6)$$

Furthermore, in view of separation of variables, we seek solution of eqns (5) and (6) in the following form

$$f(\xi,\eta) = \bar{f}(\xi)\cos\frac{nL}{R}\eta$$

$$w(\xi,\eta) = \bar{w}(\xi)\cos\frac{nL}{R}\eta, \tag{7}$$

where n denotes the number of waves in the circumferential direction during buckling. Equations (5) and (6) are thus transformed into ordinary differential equations

$$H^2\bar{f}^{(4)} - 2H\frac{dH}{d\xi}\bar{f}''' + \left[-2H^2\left(\frac{nL}{R}\right)^2 + 2\left(\frac{dH}{d\xi}\right)^2 - H\frac{d^2H}{d\xi^2}\right]\bar{f}'' + 2H\frac{dH}{d\xi}\left(\frac{nL}{R}\right)^2\bar{f}'$$
$$+\left[H^2\left(\frac{nL}{R}\right)^4 + 2\left(\frac{dH}{d\xi}\right)^2\nu\left(\frac{nL}{R}\right)^2 - H\frac{d^2H}{d\xi}\nu\left(\frac{nL}{R}\right)^2\right]\bar{f}$$
$$= \frac{12(1-\nu^2)L^2}{Rh_0}H^3\bar{w}'', \tag{8}$$

$$H^3\bar{w}^{(4)} + 6H^2\frac{dH}{d\xi}\bar{w}''' + \left[-2H^3\left(\frac{nL}{R}\right)^2 + \frac{P_0L^2}{D_0} + 6H\left(\frac{dH}{d\xi}\right)^2 + 3H^2\frac{d^2H}{d\xi^2}\right]\bar{w}''$$
$$-6H^2\frac{dH}{d\xi}\left(\frac{nL}{R}\right)^2\bar{w}' + \left[H^3\left(\frac{nL}{R}\right)^4 - 6H\left(\frac{dH}{d\xi}\right)^2\nu\left(\frac{nL}{R}\right)^2\right.$$
$$\left.-3H^2\frac{d^2H}{d\xi^2}\nu\left(\frac{nL}{R}\right)^2\right]\bar{w} + \frac{L^2}{Rh_0}\bar{f}'' = 0. \tag{9}$$

In this study, three different methods are used to obtain the classical buckling load P_{cl}. First, we evaluate the buckling load via an analytical technique, and then compare it with the results of purely numerical calculations.

3. HYBRID PERTURBATION-WEIGHTED RESIDUALS METHOD

We assume $\bar{w}(\xi)$ in the form

$$\bar{w}(\xi) = A\cos\frac{pL}{R}\xi + B\cos\frac{3pL}{R}\xi, \tag{10}$$

where p is the number of half-waves along the shell length at buckling. A and B are undetermined constants. The above buckling pattern satisfies the boundary conditions of the simple supports. The first term of the two-term approximation (10) is the exact buckling mode for the shell of constant thickness, and the second term is introduced to account for the thickness variation.

In order to solve the compatibility equation (8) for $\bar{f}$, the perturbation procedure will be employed here. To this end, $\bar{f}$ is expressed in terms of the thickness variation parameter ε as

$$\bar{f}(\xi) = f_0(\xi) + \varepsilon f_1(\xi) + \varepsilon^2 f_2(\xi) + \cdots. \tag{11}$$

Substituting eqn (11) into (8) and bearing in mind eqn (3), one has, after collecting the like terms in ε

$$f^{(4)}-2N^2f_0''+f_0-4c^2z^2\bar{w}''+\varepsilon[f_1^{(4)}-2N^2f_1''+N^4f_1-2\cos 2P\xi f_0^{(4)}$$
$$-4P\sin 2P\xi f_0'''+4N^2\cos 2P\xi f_0''-4P^2\cos 2P\xi f_0''-4PN^2\sin 2P\xi f_0'$$
$$-(2N^4+4\nu P^2N^2)\cos 2P\xi f_0+12c^2z^2\cos 2P\xi\bar{w}'']+\varepsilon^2[f_2^{(4)}-2N^2f_2''$$
$$+N^4f_2+\cos^2 2P\xi f_0^{(4)}-2\cos 2P\xi f_1^{(4)}+4P\cos 2P\xi\sin 2P\xi f_0'''-4P\sin 2P\xi f_1'''$$
$$+(-2N^2\cos^2 2P\xi+8P^2\sin^2 2P\xi+4P^2\cos^2 2P\xi)f_0''-(4P^2-4N^2)\cos 2P\xi f_1''$$
$$-4PN^2\cos 2P\xi\sin 2P\xi f_0''+4PN^2\sin 2P\xi f_1'+(N^4\cos^2 2P\xi$$
$$+8\nu N^2P^2\sin^2 2P\xi+4\nu P^2N^2\cos^2 2P\xi)f_0-(2N^4+4P^2N^2\nu)\cos 2P\xi f_1$$
$$-12c^2z^2\cos^2 2P\xi\bar{w}'']+\cdots=0, \tag{12}$$

where

$$P=\frac{pL}{R},\quad N=\frac{nL}{R},\quad z=\frac{L}{\sqrt{Rh_0}},\quad c=\sqrt{3(1-\nu^2)}. \tag{13}$$

From eqn (12), we obtain

$$\mathscr{L}(f_0)=4c^2z^2\bar{w}'' \tag{14}$$

$$\mathscr{L}(f_1)=2\cos 2P\xi f_0^{(4)}+4P\sin 2P\xi f_0'''-4N^2\cos 2P\xi f_0''+4P^2\cos 2P^2\xi f_0''$$
$$-4PN^2\sin 2P\xi f_0'+(2N^4+4\nu P^2N^2)\cos 2P\xi f_0-12c^2z^2\cos 2P\xi\bar{w}'' \tag{15}$$

$$\mathscr{L}(f_2)=-\cos^2 2P\xi f_0^{(4)}+2\cos 2P\xi f_1^{(4)}-4P\cos 2P\xi\sin 2P\xi f_0'''+4P\sin 2P\xi f_1'''$$
$$-(-2N^2\cos^2 2P\xi+8P^2\sin^2 2p\xi+4P^2\cos^2 2P\xi)f_0''+(4P^2-4N^2)\cos 2P\xi f_1''$$
$$+4PN^2\cos 2P\xi\sin 2P\xi f_0'-4PN^2\sin 2P\xi f_1'-(N^4\cos^2 2P\xi$$
$$+8\nu N^2P^2\sin^2 2P\xi+4\nu P^2N^2\cos^2 2P\xi)f_0$$
$$+(2N^4+4P^2N^2\nu)\cos 2P\xi f_1+12c^2z^2\cos^2 2P\xi\bar{w}'', \tag{16}$$

where the operator $\mathscr{L}(\cdot)$ is defined as

$$\mathscr{L}(f)=f^{(4)}-2N^2f''+N^4f. \tag{17}$$

Equations (14)–(16) can be solved analytically with the aid of the computerized symbolic algebra *Mathematica* (Wolfram, 1991) for f_0, f_1 and f_2 to yield

$$f_0=a_1\cos P\xi+a_2\cos 3P\xi$$
$$f_1=a_3\cos P\xi+a_4\cos 3P\xi+a_5\cos 5P\xi$$
$$f_2=a_6\cos P\xi+a_7\cos 3P\xi+a_8\cos 5P\xi+a_9\cos 7P\xi, \tag{18}$$

where $a_1, a_2, \ldots, a_9$ are coefficients depending on A and B, and are given in the Appendix.

Applying the weighted residuals method, namely, in our case the Boobnov–Galerkin procedure, to the equilibrium equation (9), we arrive at

$$\int_{-1/2}^{1/2}\left\{H^3\bar{w}^{(4)}+6H^2\frac{dH}{d\xi}\bar{w}'''+\left[-2H^3N^2+4\alpha cz^2+6H\left(\frac{dH}{d\xi}\right)^2+3H^2\frac{d^2H}{d\xi^2}\right]\bar{w}''\right.$$
$$-6H^2\frac{dH}{d\xi}N^2\bar{w}'+\left[H^3N^4-6H\left(\frac{dH}{d\xi}\right)^2\nu N^2-3H^2\frac{d^2H}{d\xi^2}\nu N^2\right]\bar{w}$$
$$\left.+z^2(f_0''+\varepsilon f_1''+\varepsilon^2 f_2+\cdots)\right\}\left\{\begin{matrix}\cos P\xi\\ \cos 3P\xi\end{matrix}\right\}d\xi=0, \quad (19)$$

where α is the buckling load reduction factor due to the thickness variation defined as

$$\alpha=\frac{P_0}{P_{0,\mathrm{const}}},\qquad P_{0,\mathrm{const}}=\frac{Eh_0^2}{R\sqrt{3(1-\nu^2)}} \quad (20)$$

and $P_{0,\mathrm{const}}$ is the classical buckling load of the uniform shell with constant thickness h_0.

Case A

We evaluate the classical buckling load corresponding to the buckling mode at the top of the Koiter semi-circle (Koiter, 1963). In this case, the buckling mode has the same wave numbers in both the axial and circumferential directions, and the buckling wave numbers p and n can be expressed as follows:

$$p=n=\frac{p_0}{2},\qquad p_0^2=2c\frac{R}{h_0}, \quad (21)$$

then the thickness variation pattern (3) becomes

$$h=h_0\left(1-\varepsilon\cos\frac{p_0x}{R}\right). \quad (22)$$

With this assumption, substituting eqns (11) and (18) into (19) and making some algebraic manipulations leads, when retaining only the terms up to ε^2, to the following eigenvalue problem

$$[C(\varepsilon,\alpha)]_{2\times 2}\left\{\begin{matrix}A\\ B\end{matrix}\right\}=0, \quad (23)$$

where $[C(\varepsilon,\alpha)]$ is the coefficient matrix containing the thickness variation parameter ε and the buckling load reduction factor α. The elements of matrix $[C(\varepsilon,\alpha)]$ are as follows:

$$C_{11}=P^4\left[4-4\alpha-2\varepsilon\nu+\frac{58-4\nu+13\nu^2}{25}\varepsilon^2\right]$$
$$C_{12}=C_{21}=P^4\left[-\frac{336+66\nu}{25}\varepsilon+\frac{66+300\nu+9\nu^2}{50}\varepsilon^2\right]$$
$$C_{22}=P^4\left[\frac{1412-900\alpha}{25}+\frac{1{,}571{,}010-11{,}988\nu+1377\nu^2}{21{,}125}\varepsilon^2\right]. \quad (24)$$

The requirement of vanishing of the determinant of matrix $[C(\varepsilon,\alpha)]$ results in the following equation, when the terms higher than ε^2 are neglected

$$144\alpha^2+\left(-\frac{9248}{25}+72\nu\varepsilon+\frac{-8,048,400+169,632\nu-400,968\nu^2}{21,125}\varepsilon^2\right)\alpha$$

$$+\frac{5648}{25}-\frac{2824}{25}\nu\varepsilon-\frac{1,737,952\nu-478,708\nu^2}{21,125}\varepsilon^2=0. \quad (25)$$

From eqn (25) an asymptotic expression can be obtained for the buckling load reduction factor due to the thickness variation

$$\alpha = 1-\frac{1}{2}\nu\varepsilon-\frac{(832+464\nu-23\nu^2)}{512}\varepsilon^2 \quad (26)$$

which coincides with the formula (Koiter, 1992)

$$\alpha = 1-\tfrac{1}{2}\nu\varepsilon \quad (27)$$

if the quadratic term in (26) is dropped.

Case B

We now investigate the axisymmetric buckling mode, i.e.

$$n=0, \quad p=p_0, \quad p_0^2=2c\frac{R}{h_0} \quad (28)$$

then the thickness variation pattern (3) is

$$h=h_0\left(1-\varepsilon\cos\frac{2p_0x}{R}\right). \quad (29)$$

For this case, we obtain, by retaining the terms up to ε^2, the following asymptotic expression for the buckling load reduction factor

$$\alpha = 1-\varepsilon-\frac{25}{32}\varepsilon^2 \quad (30)$$

which again coincides with Koiter's linear formula (Koiter, 1993)

$$\alpha = 1-\varepsilon \quad (31)$$

if the quadratic term is ignored.

4. SOLUTION BY FINITE DIFFERENCE METHOD

The finite difference method, which is particularly useful for the buckling problems of structures of complicated geometry or varying flexural rigidity, is used here. This method is based on the use of approximate algebraic expressions for the derivatives of unknown variables which appear in the fundamental governing equations. The following expressions of the central difference method are used to approximate the corresponding derivatives

$$\Delta f_i=\frac{f_{i+1}-f_{i-1}}{2d}, \quad \Delta^2 f_i=\frac{f_{i+1}-2f_i+f_{i-1}}{d^2},$$

$$\Delta^3 f_i=\frac{f_{i+2}-2f_{i+1}-2f_{i-1}-f_{i-2}}{2d^3}, \quad \Delta^4 f_i=\frac{f_{i+2}-4f_{i+1}+6f_i-4f_{i-1}+f_{i-2}}{d^4}, \quad (32)$$

where d is the distance between neighbouring nodal points.

Using (32), the differential equations (8) and (9) are approximated by the finite difference equations

$$
\left(\frac{G_{1i}}{d^4}-\frac{G_{2i}}{2d^3}\right)f_{i-2}+\left(-4\frac{G_{1i}}{d^4}+\frac{G_{2i}}{d^3}+\frac{G_{3i}}{d^2}-\frac{G_{4i}}{2d}\right)f_{i-1}
+\left(6\frac{G_{1i}}{d^4}-2\frac{G_{3i}}{d^2}+G_{5i}\right)f_i+\left(-4\frac{G_{1i}}{d^4}-\frac{G_{2i}}{d^3}+\frac{G_{3i}}{d^2}+\frac{G_{4i}}{2d}\right)f_{i-1}
+\left(\frac{G_{1i}}{d^4}+\frac{G_{2i}}{2d^3}\right)f_{i+2}+\frac{G_{6i}}{d^2}w_{i-1}-2\frac{G_{6i}}{d^2}w_i+\frac{G_{6i}}{d^2}w_{i+1}=0, \quad (33)
$$

$$
\frac{G_{7i}}{d^2}f_{i-1}-2\frac{G_{7i}}{d^2}f_i+\frac{G_{7i}}{d^2}f_{i+1}+\left(\frac{G_{8i}}{d^4}-\frac{G_{9i}}{2d^3}\right)w_{i-2}
+\left(-4\frac{G_{8i}}{d^4}+\frac{G_{9i}}{d^3}+\frac{G_{10i}}{d^2}-\frac{G_{11i}}{2d}\right)w_{i-1}+\left(6\frac{G_{8i}}{d^4}-2\frac{G_{10i}}{d^2}+G_{12i}\right)w_i
+\left(-4\frac{G_{8i}}{d^4}-\frac{G_{9i}}{d^3}+\frac{G_{10i}}{d^2}+\frac{G_{11i}}{2d}\right)w_{i+1}+\left(\frac{G_{8i}}{d^4}+\frac{G_{9i}}{2d^3}\right)w_{i+2}=0, \quad (34)
$$

where

$$
\begin{aligned}
&G_{1i}=[H(\xi_i)]^2, \quad G_{2i}=-2H(\xi_i)H'(\xi_i)\\
&G_{3i}=-2N^2[H(\xi_i)]^2+2[H'(\xi_i)]^2-H(\xi_i)H''(\xi_i)\\
&G_{4i}=3N^2H(\xi_i)H'(\xi_i),\\
&G_{5i}=H^2N^4+2\nu N^2[H'(\xi_i)]^2-\nu N^2H(\xi_i)H''(\xi_i)\\
&G_{6i}=-12(1-\nu^2)z^2[H(\xi_i)]^3, \quad G_{7i}=z^2\\
&G_{8i}=[H(\xi_i)]^3, \quad G_{9i}=6[H(\xi_i)]^2H'(\xi_i)\\
&G_{10i}=-2N^2H^3+4\alpha cz^2+6H(\xi_i)[H'(\xi_i)]^2+3[H(\xi_i)]^2H''(\xi_i)\\
&G_{11i}=-6N^2[H(\xi_i)]^2H'(\xi_i),\\
&G_{12i}=N^4[H(\xi_i)]^3-6\nu N^2H[H'(\xi_i)]^2-3\nu N^2[H(\xi_i)]^2.
\end{aligned} \quad (35)
$$

Here the derivatives $H'(\xi_i)$ and $H''(\xi_i)$ are evaluated analytically. By subdividing the shell length domain $(-L/2, L/2)$ into M equal segments and applying eqns (33) and (34) to each nodal point, points near the ends of the shell are influenced by the boundary conditions. Here we consider the case of simply-supported boundary conditions, namely

$$
w_0=w_0''=f_0=f_0''=w_M=w_M''=f_M=f_M''=0, \quad (36)
$$

or in view of eqn (32)

$$
w_{-1}=w_1, \quad f_{-1}=f_1, \quad w_{M+1}=w_{M-1}, \quad f_{M+1}=f_{M-1}. \quad (37)
$$

Thus, we establish a system of simultaneous algebraic equations,

$$
[C(\xi_i,\alpha)]_{(2M+2)\times(2M+2)}\{\delta\}_{(2M+2)\times(2M+2)}=0, \quad (38)
$$

where $[C(\xi_i,\alpha)]$ is the coefficient matrix, whose elements depend on the shell geometry, nodal point coordinates, elastic constants as well as the unknown buckling load reduction

Table 1. Comparison of buckling loads derived via different methods for Case A ($\nu = 0.3$)

ε	Asymptotic formula: Koiter's formula, eqn (27)	Asymptotic formula: Second-order approximation, eqn (26)	Shooting method	Finite difference
0.0	1.0	1.0	1.0	1.0
0.01	0.999	0.998	0.999	0.998
0.05	0.993	0.988	0.989	0.988
0.10	0.985	0.966	0.967	0.966
0.15	0.975	0.935	0.939	0.938

factor $\alpha\{\delta\}$ represents a column vector containing the unknown values of functions of w and f at the nodal points.

Setting the determinant of $[C(\xi_i, \alpha)]$ equal to zero gives the approximate value for the classical buckling load reduction factor α or the classical buckling load which improves in accuracy with an increase in the number of subdivided segments. In the implementation of this process, the classical buckling load reduction rate α is sought through iterations.

5. SOLUTION BY GODUNOV–CONTE SHOOTING METHOD

The differential equations (8) and (9), together with the simply-supported boundary conditions ($\bar{f} = \bar{f}'' = \bar{w} = \bar{w}'' = 0$ at the ends of the shell), can be solved for the classical buckling load by use of the shooting method. However, as pointed out by Grigolyuk *et al.* (1971), in the problem of buckling of cylindrical shells, when the non-dimensional parameter $L(1-\nu^2)^{0.25}(Rh)^{-0.5}$ exceeds ten, the coefficient matrix of the algebraic equations, from which the missing initial conditions are solved, becomes too ill-conditioned, which may lead to the loss of accuracy or even completely incorrect results. In the present study the modified version of the shooting method, known as the Godunov–Conte method (Elishakoff and Charmats, 1977), is employed. It utilizes the Gram–Schmidt orthogonalization procedure during the integration steps to prevent the ill-conditioning problem so that more accurate results could be obtained than those furnished by the ordinary shooting method.

6. NUMERICAL RESULTS AND DISCUSSION

The results for the classical buckling load reduction α from the above three methods are given in the Tables 1 and 2 for different values of the thickness variation parameter ε.

A very good match between the results from different methods is shown up to the value $\varepsilon = 0.05$. The increasingly larger difference is observed between the results of the first-order approximation given by eqn (27) [or eqn (31)] and those of numerical solutions as ε becomes larger. While the first-order asymptotic approximate formula may not be sufficiently accurate as ε reaches 0.1, the second-order asymptotic formula (26) [or (30)] retains a good accuracy even for ε as large as 0.15. Thus, owing to their higher accuracy, eqns (26) and (30) can be used to obtain a sufficiently good estimate of the buckling load reduction factor due to the thickness variation.

Table 2. Comparison of buckling loads derived via different methods for Case B ($\nu = 0.3$)

ε	Asymptotic formula: Koiter's formula eqn (31)	Asymptotic formula: Second-order approximation, eqn (30)	Shooting method	Finite difference
0.0	1.0	1.0	1.0	1.0
0.01	0.990	0.990	0.990	0.990
0.05	0.950	0.948	0.949	0.948
0.10	0.900	0.892	0.895	0.894
0.15	0.850	0.832	0.837	0.836

The above results also show that the effect of certain types of thickness variation on buckling load deserves special attention. Although the thickness variation pattern akin to the classical buckling mode (Case A) may have a remarkable effect on the classical buckling load (the classical buckling load is decreased by over 6% when $\varepsilon = 0.15$), the most detrimental effect of thickness variation occurs when the wave number of the axisymmetric thickness variation is twice that of the classical buckling mode (Case B). In this situation, even if the amplitude of the thickness variation is as small as 0.1, the thickness variation reduces the buckling load by 10% from its counterpart of the shell with constant thickness. When $\varepsilon = 0.15$, the classical buckling load is decreased by over 15%. Thus, in the absence of initial geometric imperfection, this particular kind of thickness variation may constitute the most important factor in the buckling load reduction. The study of the effect of thickness variation in shells made of composite materials is underway and will be reported elsewhere.

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APPENDIX

$$a_1 = -\frac{4c^2z^2P^2}{(P^2+N^2)^2}A, \quad a_2 = -\frac{36c^2z^2P^2}{(9P^2+N^2)^2}B$$

$$a_3 = \frac{1}{(N^2+P^2)^2}[(a_1+a_2)N^4+(4a_1+24a_2)N^2P^2+(a_1+117a_2)P^4+(2va_1+2va_2)N^2P_2+(6A+54B)c^2P^2z_2]$$

$$a_4 = \frac{a_1N^4-3a_1P^4+2va_1N^2P^2+6c^2P^2z^2A}{(N^2+9P^2)^2}$$

$$a_5 = \frac{a_2N^4+12a_2N^2P^2+9a_2P^4+2va_2N^2P_2+54c^2P^2z^2B}{(N^2+25P^2)^2}$$

$$a_6 = -\frac{1}{4(N^2+P^2)^2}[(2a_1+a_2-4a_3-4a_4)N^4+(-22a_1+225a_2-4a_3-468a_4)P^4 \\ +(4a_1+30a_2-16a_3-96a_4+24va_1-4va_2-8va_4)N^2P^2+(24A+108B)c^2P^2z^2]$$

$$a_7 = -\frac{1}{4(N^2+9P^2)^2}[(a_1+2a_2-4a_3-4a_5)N^4+(9a_1-54a_2+12a_3-3300a_5)P^4 \\ +(6a_1+36a_2-240a_5-4va_1+24va_2-8va_3-8va_5)N^2P^2+(12A+216B)c^2P^2z^2]$$

$$a_8 = -\frac{1}{4(N^2+25P^2)^2}[(a_1-4a_4)N^4+(a_1-36a_4)P^4-(2a_1+48a_4+4va_1+8va_4)N^2P^2+12c^2P^2z^2A]$$

$$a_9 = -\frac{1}{4(N^2+49P^2)^2}[(a_2-4a_5)N^4+(9a_2-1300a_5)P^4+(6a_2-160a_5-4va_2-8va_5)N^2P^2+108c^2P^2z^2B].$$